THE CHINESE UNIVERSITY OF HONG KONG
MATH3270B
L-Transform SOLUTION

1. By definition, we have

$$\begin{aligned} L[f] &= \int_0^\infty e^{-(s-1)t} t^n dt \\ &= -\frac{1}{s-1} t^n e^{-(s-1)t}|_0^\infty + \int_0^\infty \frac{1}{s-1} e^{-(s-1)t} n t^{n-1} dt (integral\ by\ parts) \\ &= \int_0^\infty \frac{1}{s-1} e^{-(s-1)t} n t^{n-1} dt \\ &= ... = (use\ integral\ by\ parts\ n\ times) \\ &= \int_0^\infty n! \frac{1}{(s-1)^n} e^{-(s-1)t} dt \\ &= \frac{n!}{(s-1)^{n+1}}. \end{aligned} \tag{1}$$

2. We denote $Y(s) = L[f(t)]$ and use laplace transform on both sides,

$$(s^2 + 2s + 7)Y(s) = \frac{1}{s-1} + s + 3, \tag{2}$$

which means $Y(s) = \frac{1}{(s-1)(s^2+2s+7)} + \frac{(s+3)}{s^2+2s+7}$. Use partial fraction, we deduce that

$$Y(s) = \frac{1}{8}\left(\frac{1}{s-1} + \frac{s+1}{(s+1)^2 + (\sqrt{6})^2} + -\frac{2}{\sqrt{6}} \frac{\sqrt{6}}{(s+1)^2 + (\sqrt{6})^2}\right) + \frac{s+1}{(s+1)^2 + (\sqrt{6})^2} + \frac{2}{\sqrt{6}} \frac{\sqrt{6}}{(s+1)^2 + (\sqrt{6})^2}, \tag{3}$$

Now the inverse L-transform of R.H.S is $\frac{1}{8}(e^t + e^{-t}\cos(\sqrt{6}t) - \frac{2}{\sqrt{6}} e^{-t}\sin(\sqrt{6}t)) + e^{-t}\cos(\sqrt{6}t) + \frac{2}{\sqrt{6}} e^{-t}\sin(\sqrt{6}t)$, which is sol of that O.D.E.

3. This pro is similar to problem 2. So I omit the process here and just give the solution is $y(t) = \frac{5}{21}\sin(2t) + \frac{1}{2}\sin(2t) + \cos(2t) - \frac{2}{21}\sin(5t)$.

4. (a) We assume $F(s) = \frac{As+B}{(s-2)^2} + \frac{C}{s+3}$. It's easy to deduce that $A = \frac{7}{25}, B = \frac{51}{25}$, $C = \frac{-7}{25}$. Hence,

$$F(s) = \frac{7}{25}\frac{1}{s-2} + \frac{13}{5}\frac{1}{(s-2)^2} - \frac{7}{25}\frac{1}{s+3}. \tag{4}$$

Therefore, the inverse transform is $y(t) = \frac{7}{25}e^{2t} + \frac{13}{10}te^{2t} - \frac{7}{25}e^{-3t}$.

(b) This problem is similar to problem 4.1. Use partial fraction, we deduce that

$$F(s) = \frac{1}{16}(\frac{-3s-4}{s^2} + \frac{3s+13}{s^2+3s-4}). \quad (5)$$

Hence, the inverse transform is $\frac{1}{16}(-3-4t) + \frac{1}{16}(\frac{16}{5}e^t - \frac{1}{5}e^{4t})$.